Solutions

Practice Exam 2
Chapter 2 and Sections 3.1-3.2

Treat this like an exam. Answer the following questions. *You must show your work to receive full credit.*

1. Let $A = \{6, 7, 8\}$ and $B = \{1, 3, 5, 7, 9, 11\}$, and suppose the universal set is $U = \{1, 2, \ldots, 11\}$. List all the elements in the following sets.

(a) $A \cap B' = \{6, 8\}$

(b) $(A \cap B)' = \{1, 2, \ldots, 6, 8, 9, 10, 11\}$

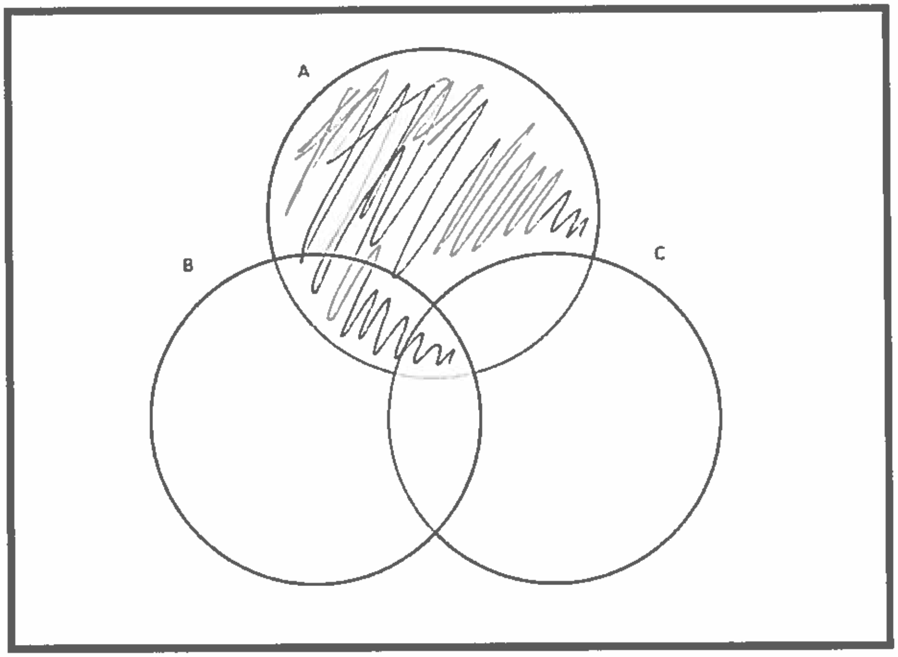
(c) $(A \cup B)' \times A = \{2, 4, 10\} \times \{6, 7, 8\} = \{(2,6), (2,7), (2,8), (4,6), (4,7), (4,8), (10,6), (10,7), (10,8)\}$

(d) $P(A \backslash B) =$
$\{\emptyset, \{6\}, \{8\}, \{6, 8\}\}$

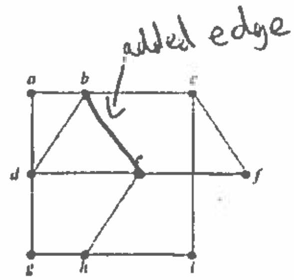
2. List all of the elements $S \in P(\{1, 2, 3, 4\})$ such that $|S| = 3$.

$\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$

3. In the venn diagram below, shade the area corresponding to $A \cap (B \cup C')$.



4. Consider the following graph.



(a) (4 points) This graph does not have an Euler path (a path which uses every edge exactly once). Explain why. Make sure to reference any theorem that you use.

(b) (4 points) Without adding new vertices, add a single edge to the graph so that the new graph will have an Euler path. In indicate the new edge by drawing it on the graph.

(a) Edges b, e and h all have degree 3 so the Theorem of Euler tells us no Euler path is possible.

(b)

5. Consider the following sets.

$G=$ the set of all good citizens. $C=$ the set of all charitable people. $P=$ the set of all polite people.

Express the statement. "Everyone who is charitable and polite is a good citizen," in the language of set theory.

$C \cap P \subseteq G$

6. Define a function $f : \mathbb{R} \to [0, \infty)$ by the formula $f(x) = x^2$.

(a) Is f one-to-one? Prove or disprove.

(b) Is f onto? Prove or disprove.

(a) Not one-to-one. because $f(-1)=1=f(1)$.

(b) Yes Onto. For any $x \in [0,\infty)$, $f(\sqrt{x})=x$.

7. Consider the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 + 1$ and $g(x) = 2x + 1$.

(a) Find $f \circ g$.

(b) Find $g \circ f$.

(a) $f \circ g(x) = f(2x+1) = (2x+1)^3 + 1$

(b) $g \circ f(x) = g(x^3+1) = 2(x^3+1)+1 = 2x^3+2$.

Let $n \in \mathbb{N}$ be a positive integer. Recall the relation on $\mathbb{Z}$ for modular arithmetic defined by $a \equiv b \mod n$ whenever $n|(a-b)$.

8. Find the equivalence class of -1 modular 4.

$$[-1] = \{k \in \mathbb{Z} \mid k = 4j - 1 \text{ for some } j \in \mathbb{Z}\}.$$

9. Show that the above relation is an equivalence relation.

Reflexive: For all $a \in \mathbb{Z}$ $n|(a-a)$ so $a \equiv a \mod n$. ✓

Transitive: Suppose $n|a-b$ and $n|b-c$. Then $a-b = nk_1$ and $b-c = nk_2$ for some integers k_1 and k_2.

Thus $a-c = (a-b)+(b-c) = nk_1 + nk_2 = n(k_1+k_2)$. So $n|a-c$. ✓

Symmetric: Suppose $n|a-b$. Then $a-b = nk$ for some $k \in \mathbb{Z}$.

Thus $b-a = n\cdot(-k)$ and $n|b-a$. ✓

10. ~~Show that the above relation is a partial ordering.~~ Show that the relation D given by aDb if $a|b$. is a partial ordering on.

~~Reflexive and Transitive proved above.~~

~~Antisymmetric: Suppose $n|a-b$ and $n|b-a$.~~

~~Then $a-b=nk_1$ and $b-a=nk_2$. Thus~~

Reflexive: $a = a\cdot 1$ so $a|a$ ✓

Transitive: If $a|b$ and $b|c$, then $b = a\cdot k_1$ and $c = b\cdot k_2$ for some $k_1, k_2 \in \mathbb{Z}$

Thus $c = a\cdot(k_1 k_2)$ and $a|c$ ✓

Antisymmetric: Suppose $a|b$ and $b|a$. Then $a = bk_1$ and $b = ak_2$ for some $k_1, k_2 \in$

Then $a = bk_1 = ak_2k_1$. So $k_2k_1 = 1$ and thus $k_1 k_2 = 1$ and $a = b$.

11. Let P be defined by

$$P(n) = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot P(n-1) & \text{if } n > 0. \end{cases}$$

(a) Use the recursive formula above to compute $P(4)$.

(b) Use induction to verify that $P(n) = n!$, where

$$n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdots (n-1) \cdot n$$

and by convention $0! = 1$.

$P(1)$

(a) $P(0) = 1$, $P(2) = 2$, $P(3) = 6$, $P(4) = 24$

(b) Base Case: $P(0) = 0! = 1$ by convention.

Inductive Step: Suppose $P(n) = n!$ for some $n \geq 0$.

Then $P(n+1) = (n+1)P(n) = (n+1) \cdot n!$

$= 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n \cdot (n+1) = (n+1)!$

So $P(n) = n!$ for all $n \geq 0$.

12. Suppose there is a zombie apocalypse starting today with 1 zombie. Each day, every zombie will create two more zombies.

(a) Assuming that no zombies are destroyed, calculate the number of zombies $Z(5)$ on day 5.

(b) Find a recurrence relation for the number of zombies $Z(n)$ on day n.

(c) Guess a closed-form for your recurrence relation.

(d) Verify that your guess is correct using induction.

(a) $Z(0)=1,\ Z(1)=3,\ Z(2)=9,\ Z(3)=27$

$Z(4)=81,\ Z(5)=243, \ldots$

(b) $Z(n)=\begin{cases} 1, & n=0 \\ 3\cdot Z(n-1), & n>0 \end{cases}$

(c) $Z(n)=3^n$

(d) <u>Base Case:</u> $Z(0)=3^0=1$ ✓

<u>Inductive Step:</u> Suppose $Z(n)=3^n$ for some $n\geq 0$.

Then $Z(n+1)=3\cdot 3^n=3^{n+1}$ ✓

So $Z(n)=3^n$ for all $n\geq 0$.

13. Let A and B be sets. Prove that $A \subseteq A \cup B$.

Let $x \in A$. Then $(x \in A) \vee (x \in B)$ by the logical addition inference rule so $x \in A \cup B$

14. Consider the following sets.

$A := \{n \in \mathbb{Z} | n = 4k + 1 \text{ for some } k \in \mathbb{Z}\}$.
$B := \{n \in \mathbb{Z} | n = 4k + 3 \text{ for some } k \in \mathbb{Z}\}$.
$E := \{n \in \mathbb{Z} | n = 2k \text{ for some } k \in \mathbb{Z}\}$.
$O := \{n \in \mathbb{Z} | n = 2k + 1 \text{ for some } k \in \mathbb{Z}\}$.

Prove the following set relationships.

(a) $A \subseteq O$.
(b) $B' \not\subseteq E$.
(c) $A' \cap B' = E$.
(d) $A \cup B = O$.
(e) $B = O \cap A'$.

(a) Let $n \in A$. Then $n = 4k+1$ for some $k \in \mathbb{Z}$. So $n = 2(2k)+1$ and $n \in O$. So $A \subseteq O$.

(b) $1 \in B'$ and $1 \notin E$. So $B' \not\subseteq E$.

(c) $A' \cap B' = (A \cup B)'$. Since $O' = E$, this is equivalent to showing (d).

(d) By (a) $A \subseteq O$. Similarly, $B \subseteq O$. So $A \cup B \subseteq O$.
Let $n \in O$. Then $n = 2k+1$ for some $k \in \mathbb{Z}$.
Since $E \cup O = \mathbb{Z}$, $k \in E$ or $k \in O$.
If $k \in E$, then $k = 2j$ for some $j \in \mathbb{Z}$ and $n = 4j+1 \in A$.
If $k \in O$, then $k = 2j+1$ for some $j \in \mathbb{Z}$ and $n = 4j+3 \in B$. So $O \subseteq A \cup B$.

(e) This is ~~equivalent~~ an identical proof to that of Example 5. Part 2 of the Section 2.2 worksheet.